James Casey

Exploring Curvature

With 141 Illustrations

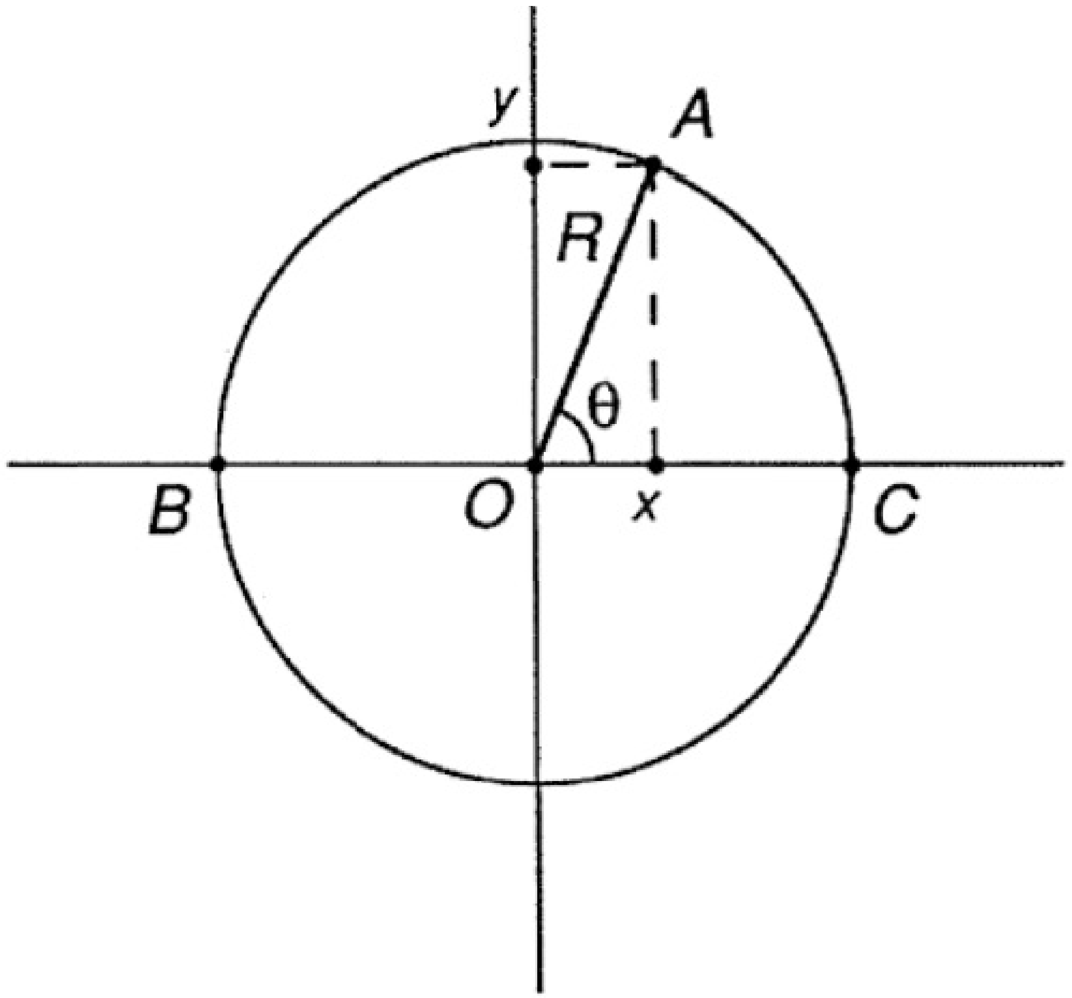
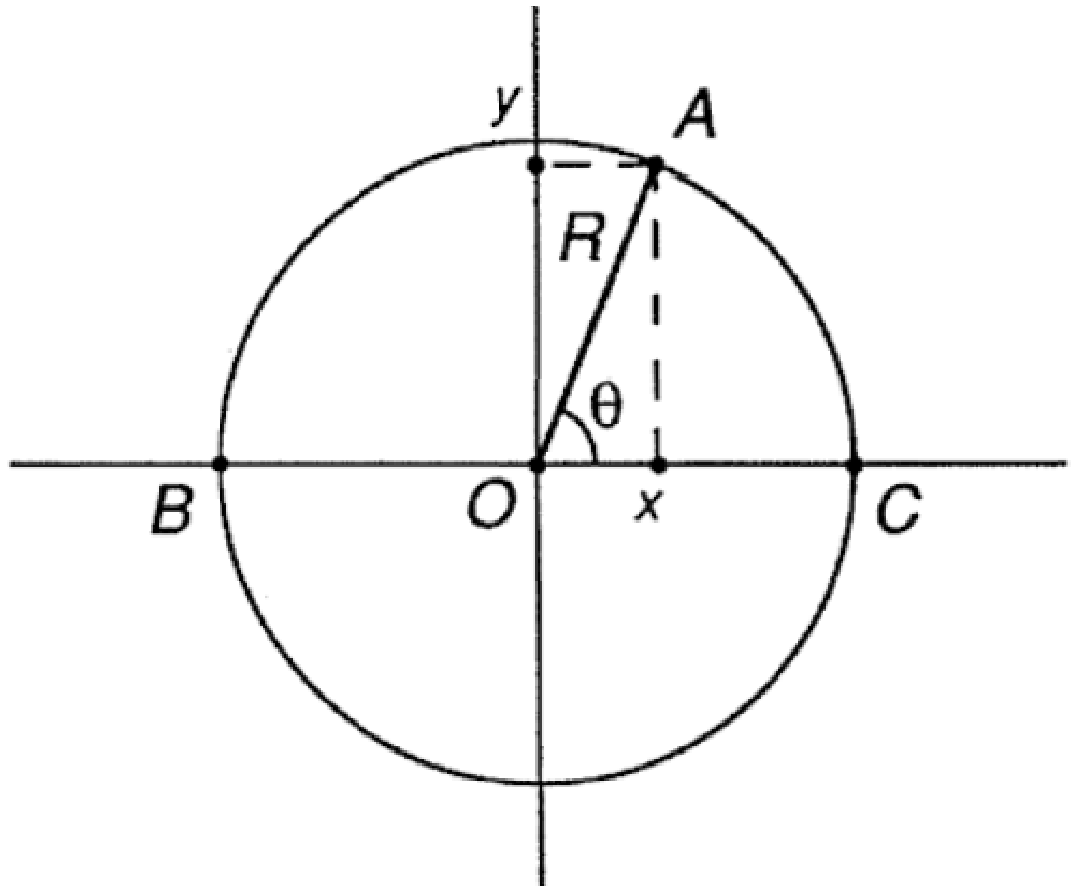


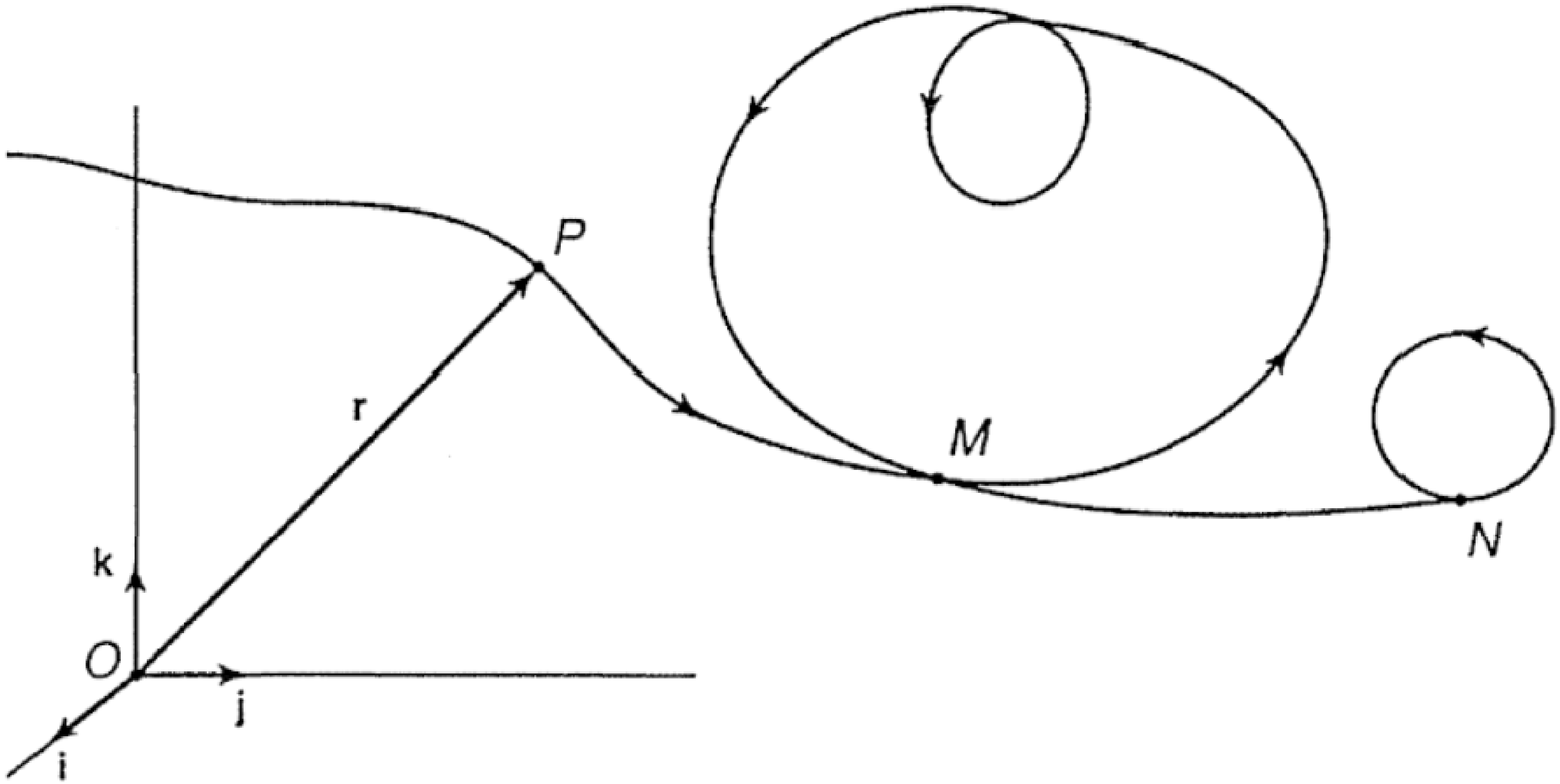
Figure 19 The circle

$$y = \pm\sqrt{R^2 - x^2} \,. \qquad (4.7)$$

The latter equation does not represent a function, because there exist values of x which when substituted into Equation (4.7) yield two, and not one, value of y. Even if there was only one value of x that did this, it

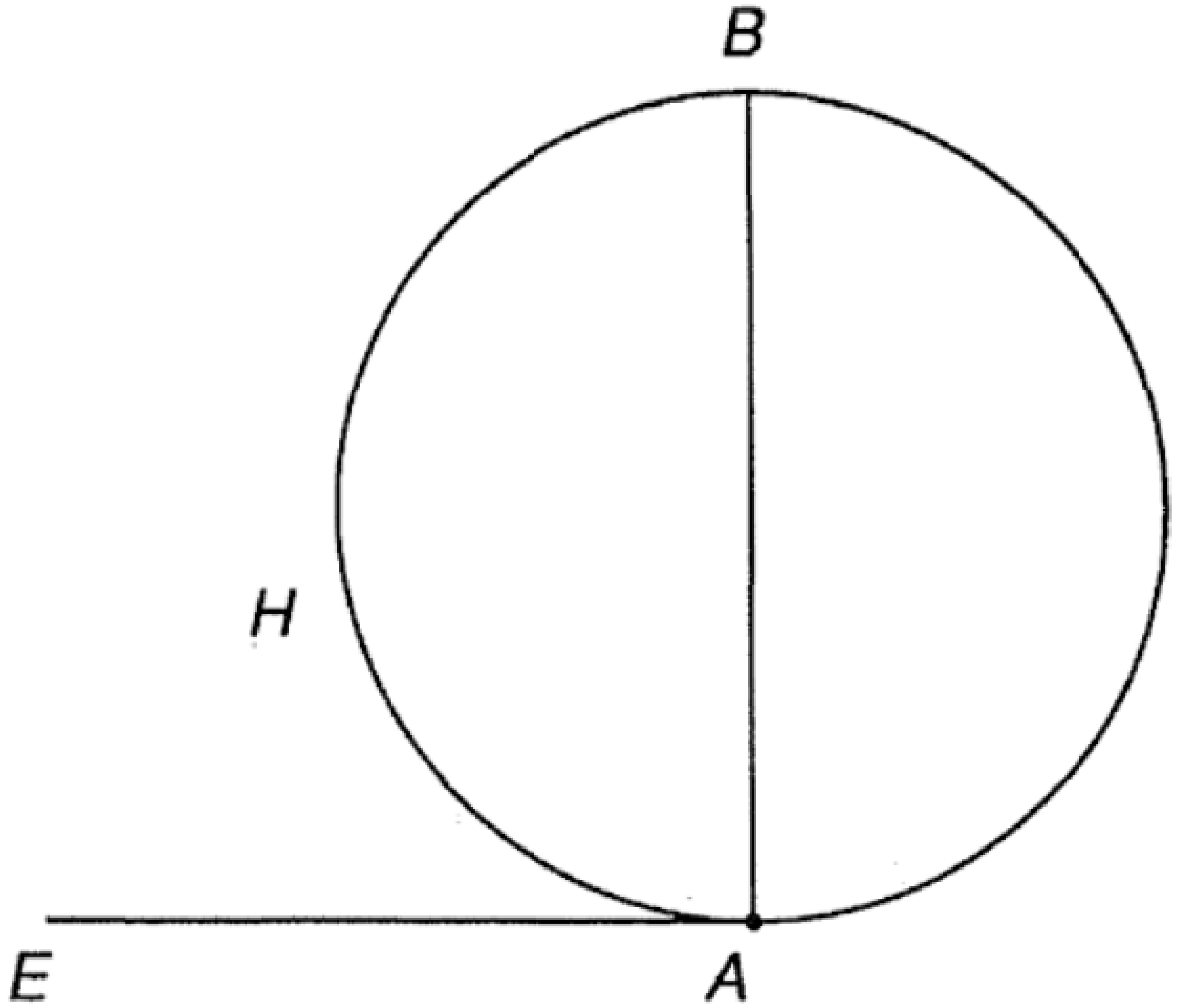


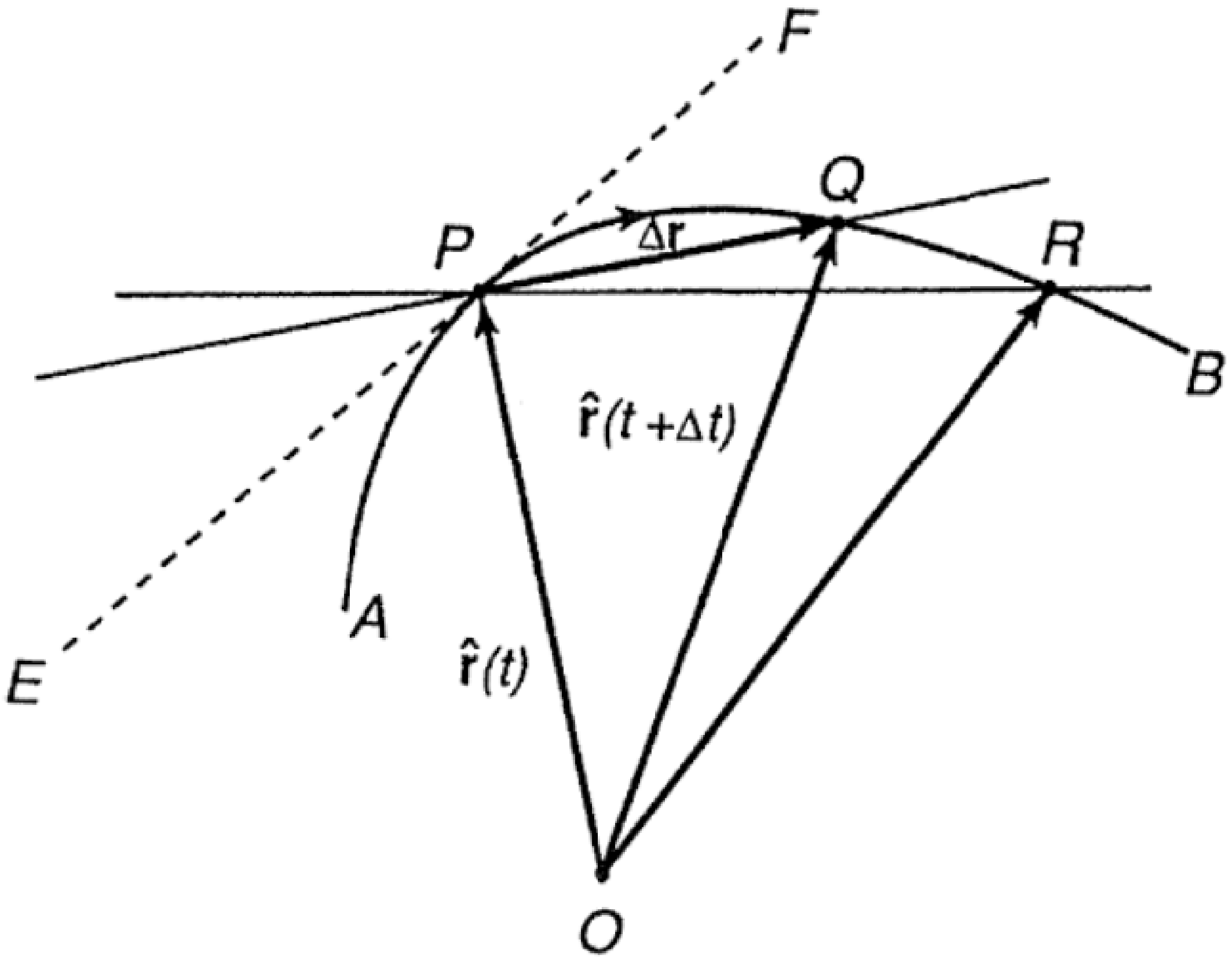
The situation encountered in Example 9 is typical: for general planar curves, a single function taking x-values into y-values would not suffice. The curve in Fig. 21 another example. In such cases, the kinematical and parametrical descriptions become very useful. Returning to the circle,



$$x = \hat{x}(t)\,, \quad y = \hat{y}(t)\,, \quad z = \hat{z}(t)\,. \qquad (7.1)$$

$$\begin{aligned} \mathbf{r} &= \hat{x}(t)\,\mathbf{i} + \hat{y}(t)\,\mathbf{j} + \hat{z}(t)\,\mathbf{k} \\ &= \hat{\mathbf{r}}(t)\,, \end{aligned} \qquad (7.2)$$


B
H
E
A


F
Q
P
Δr
R
B
r̂(t +Δt)
A
E
r̂(t)
O

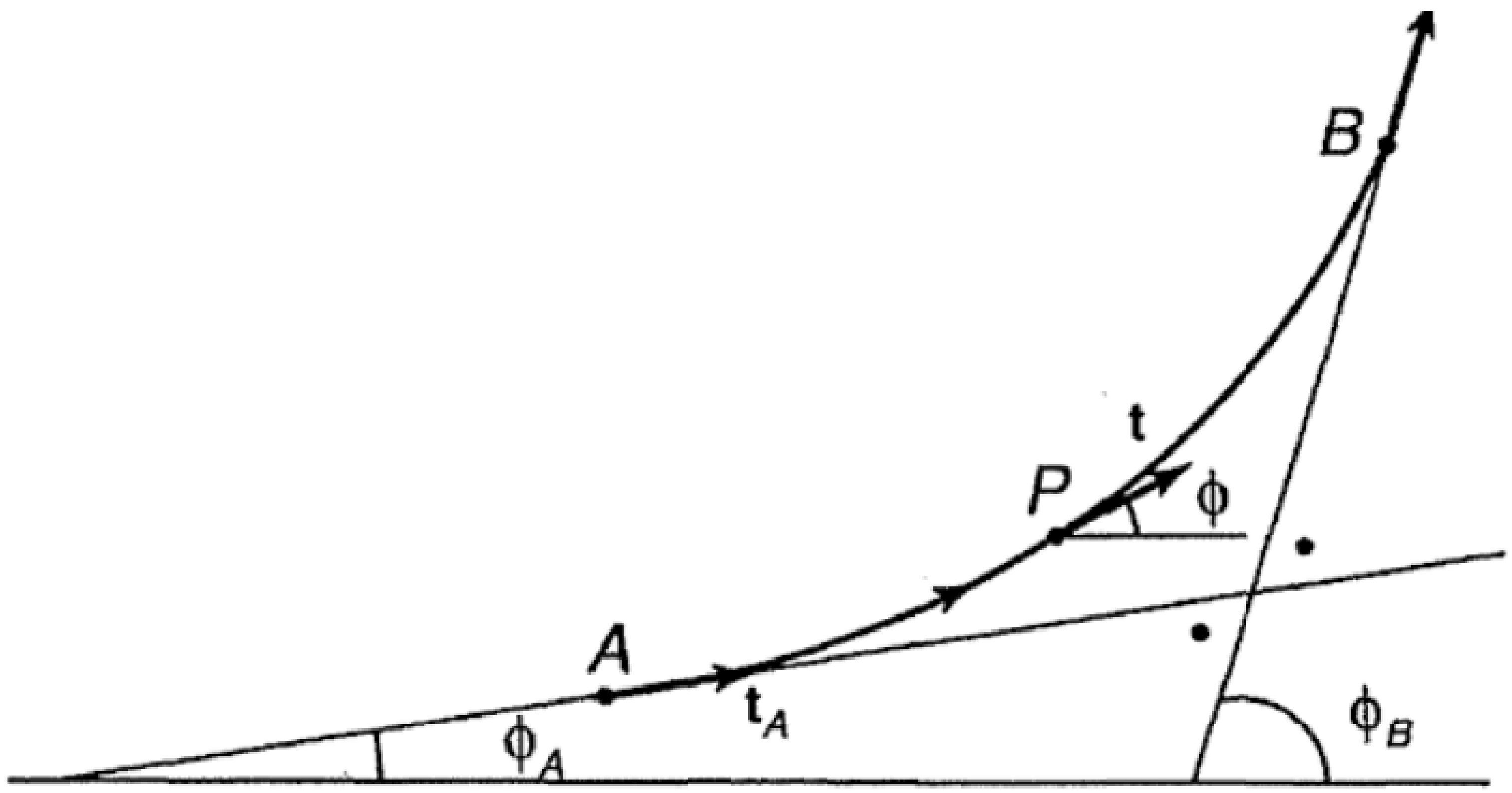


Figure 62 Change in direction of curve

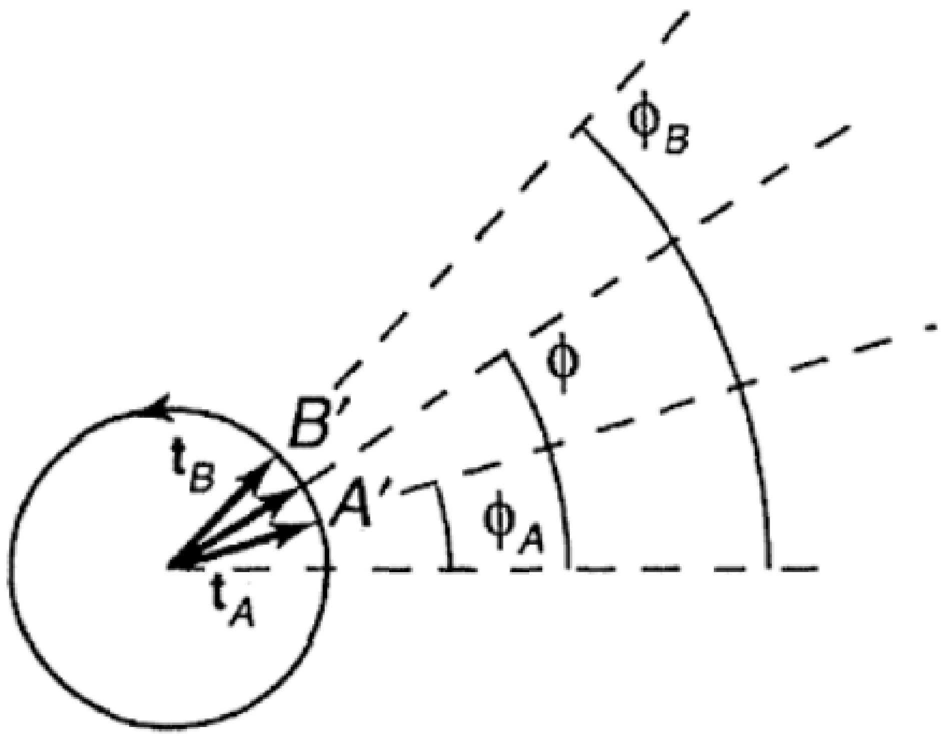
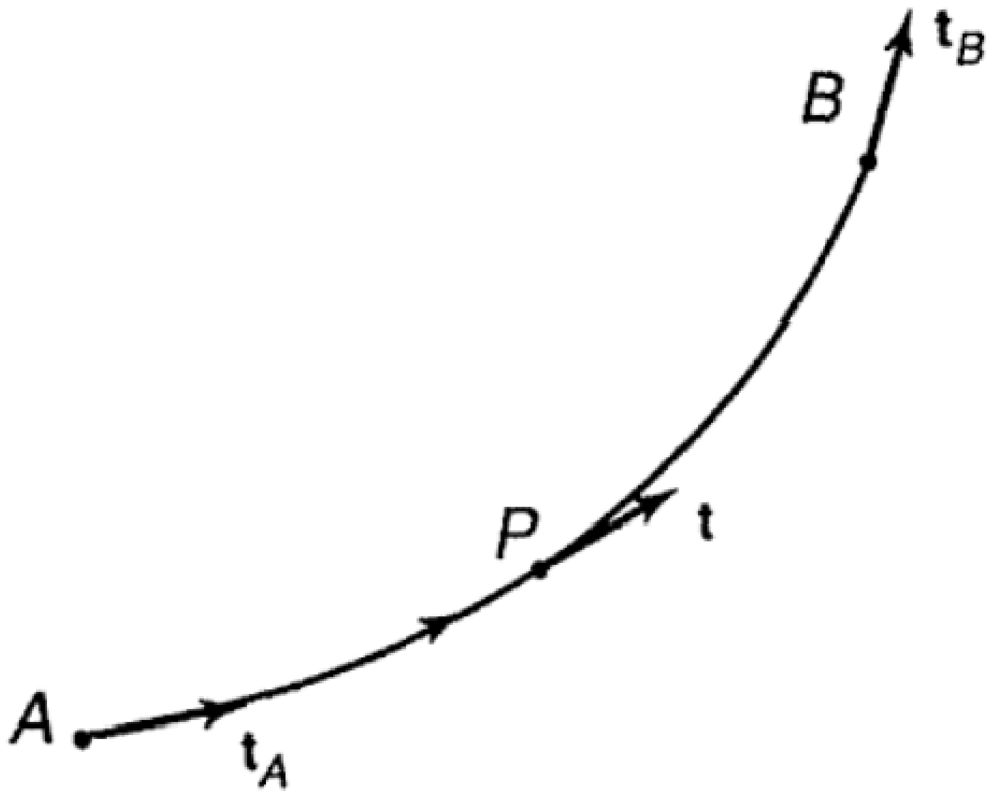
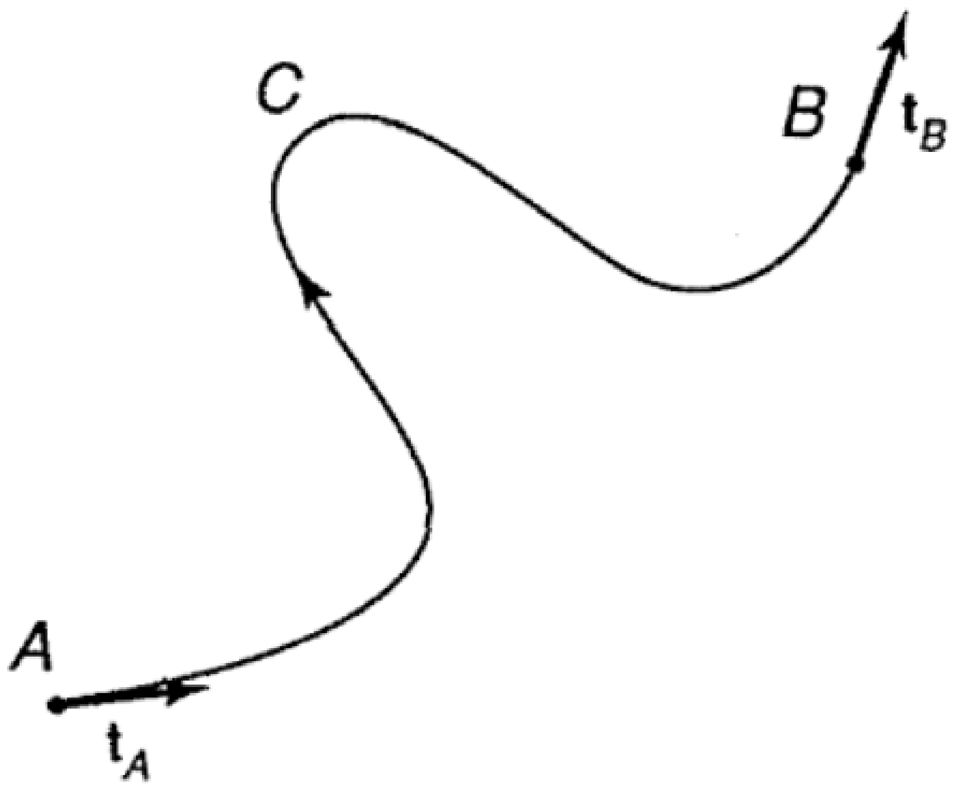


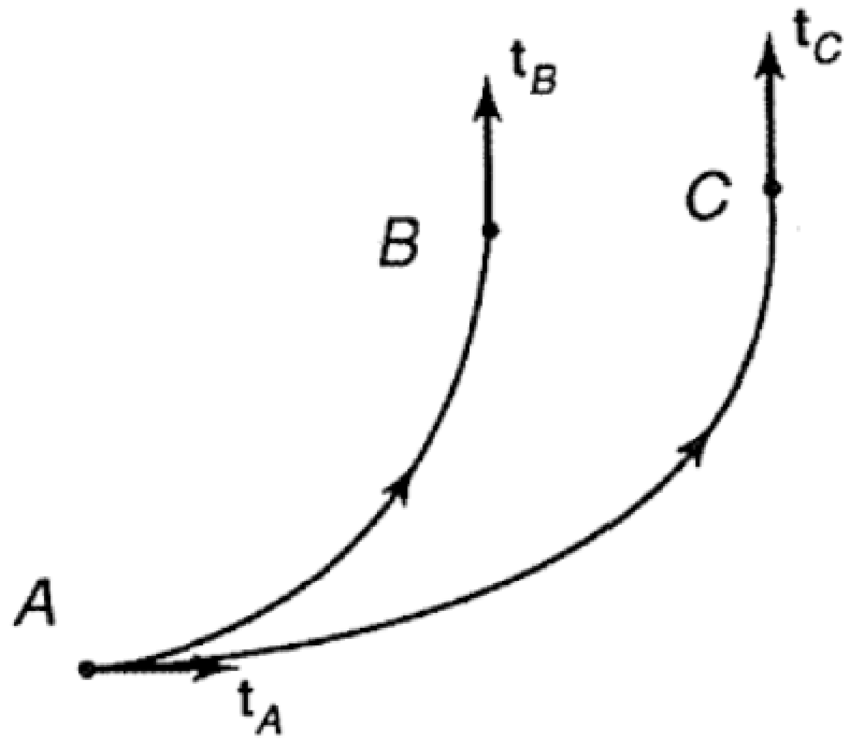
Figure 63 Auxiliary unit circle

$$\textit{total curvature of } AB = \phi_B - \phi_A \;. \qquad (10.1)$$

CURVE // AVERAGE CURVATURE





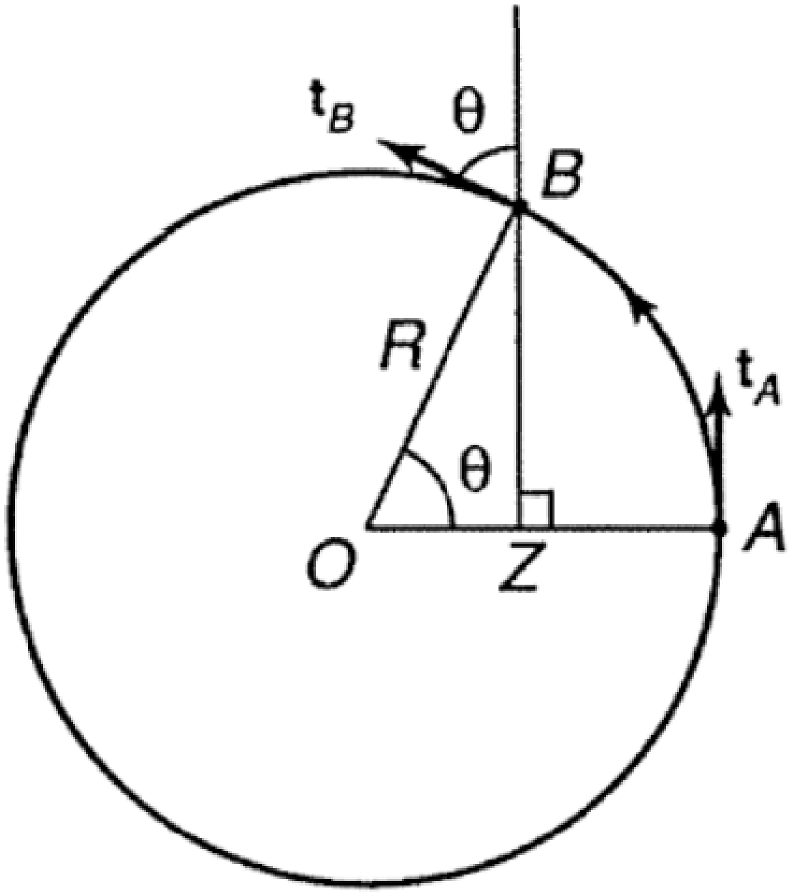


t
B
t
C
B
C
A
t
A

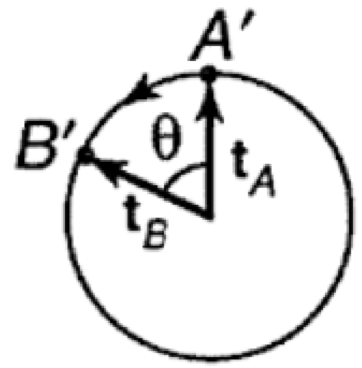
"curviness" of the various curves in Figs. 66 and 67 is to compare their total curvatures *over equal lengths* of the curves. Thus, if s_A and s_B are the arc lengths corresponding to two distinct points A and B, respectively, we define the *average curvature* of the arc AB to be

$$\kappa_{avg} = \frac{\phi_B - \phi_A}{s_B - s_A} \quad . \tag{10.2}$$

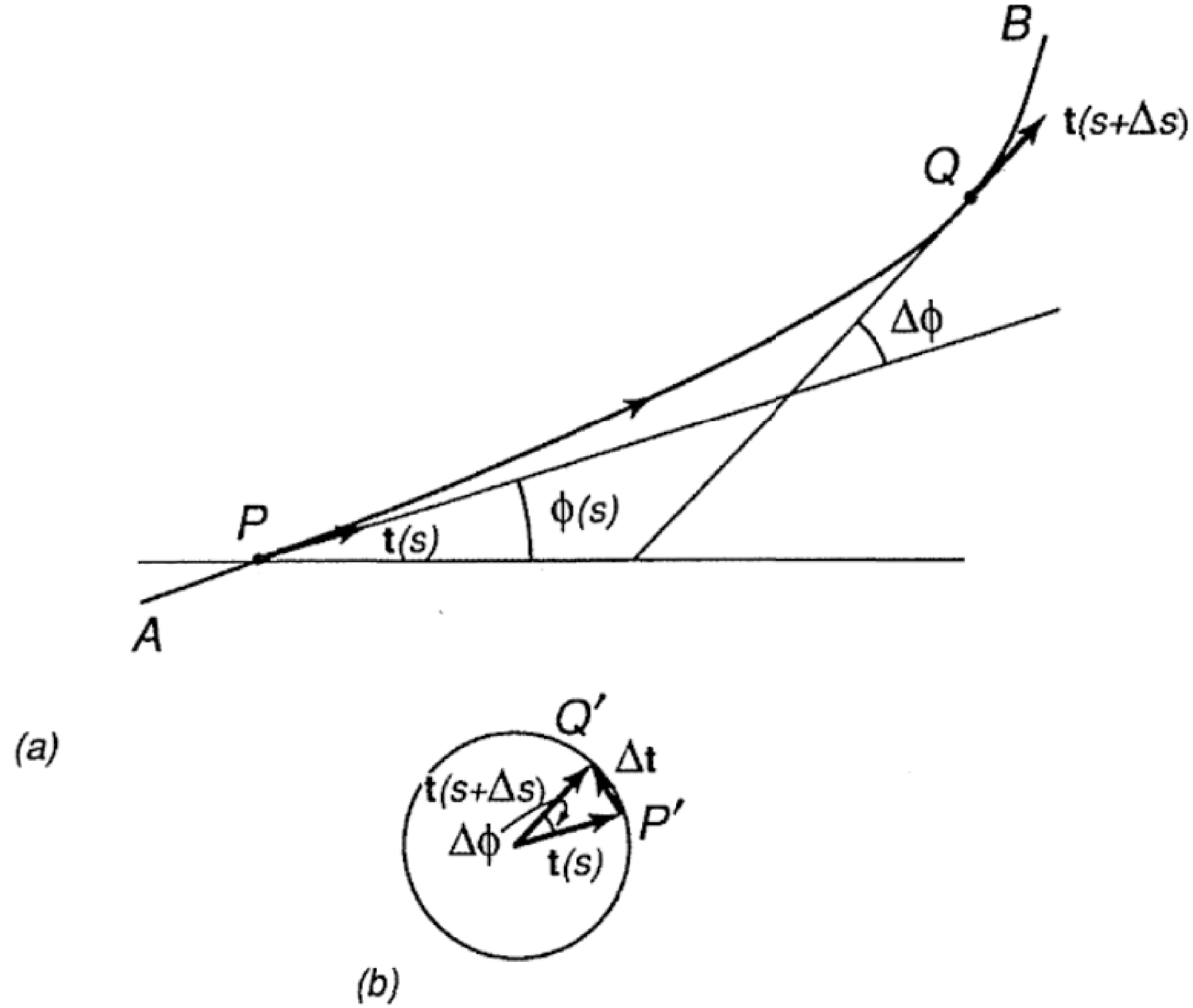
We note that for an arc of a circle subtending an angle θ, the average curvature is $\theta / R\theta$, or $1/R$.



(a)



(b)



(a)

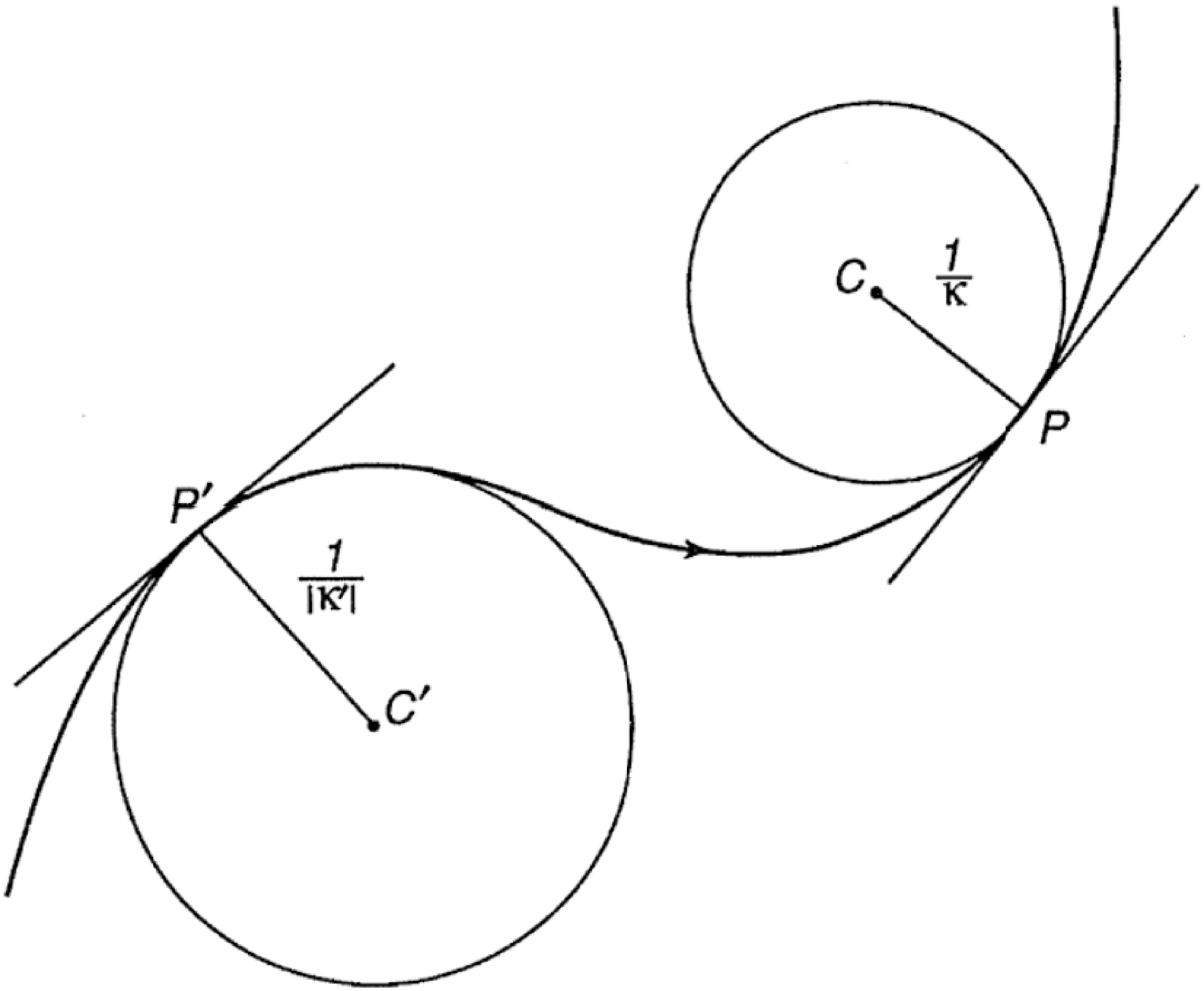
(b)

at an angle $\phi(s + \Delta s)$ to the horizontal direction. The angle through which the tangent rotates as one moves along the curve from P to Q is $\phi(s+\Delta s) - \phi(s)$, which we shall write as $\Delta\phi$ (see Fig. 68a). The tangent vectors $\mathbf{t}(s)$ and $\mathbf{t}(s + \Delta s)$, and the angle $\Delta\phi$, are shown again in the auxiliary unit circle in Fig. 68b. In accordance with Equation (10.2), the average curvature of the arc PQ is $\Delta\phi/\Delta s$. Taking a hint from calculus, we let Q lie closer and closer to P. Then, if the quotient $\Delta\phi / \Delta s$ tends to a limiting value κ, we define this to be the *curvature of the curve at* P. Thus, we have

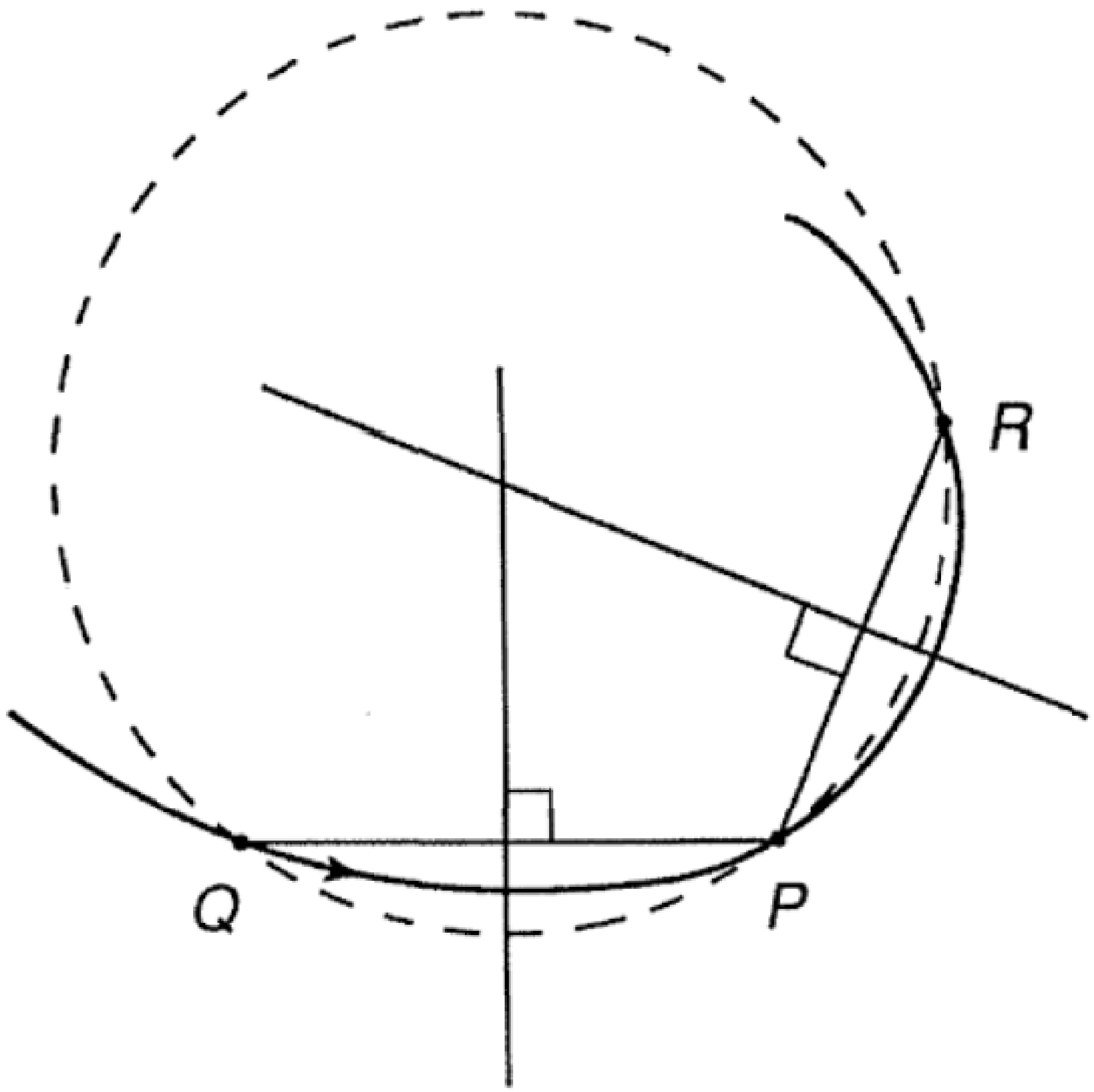
$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds} , \qquad (10.3)$$

i.e., κ is the derivative of the angle ϕ, regarded as a function of arc length. For example, in the case of a circle of radius R, since $\Delta s = R\,\Delta\phi$, we have

$$\begin{aligned} \kappa &= \lim_{\Delta s \to 0} \frac{\Delta s / R}{\Delta s} = \frac{1}{R} \lim_{\Delta s \to 0} \frac{\Delta s}{\Delta s} \\ &= \frac{1}{R} . \end{aligned} \qquad (10.4)$$


C
1
κ
P
P′
1
|κ′|
C′

CURVE // CIRCLES OF CURVATURE



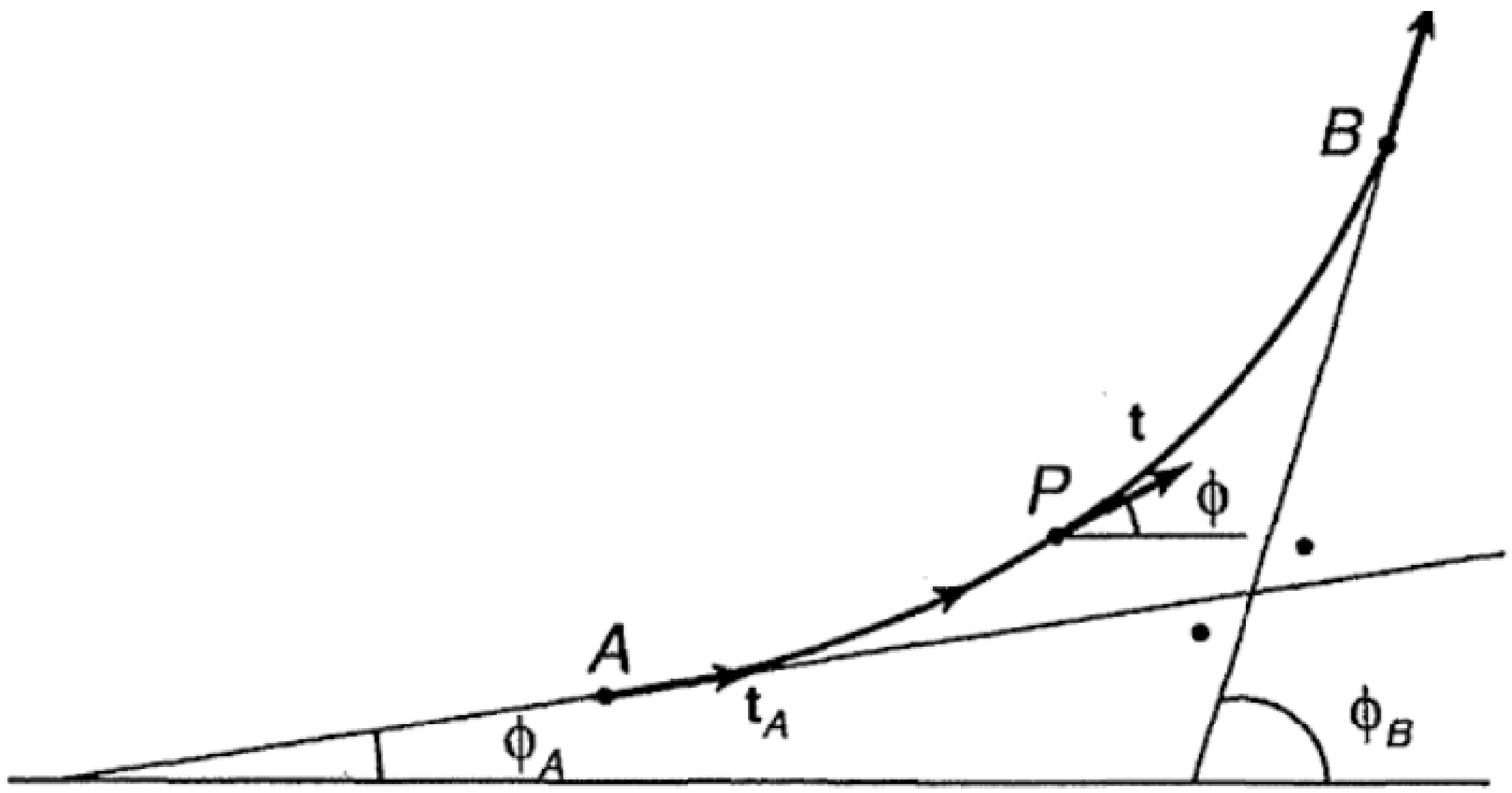


Figure 62 Change in direction of curve

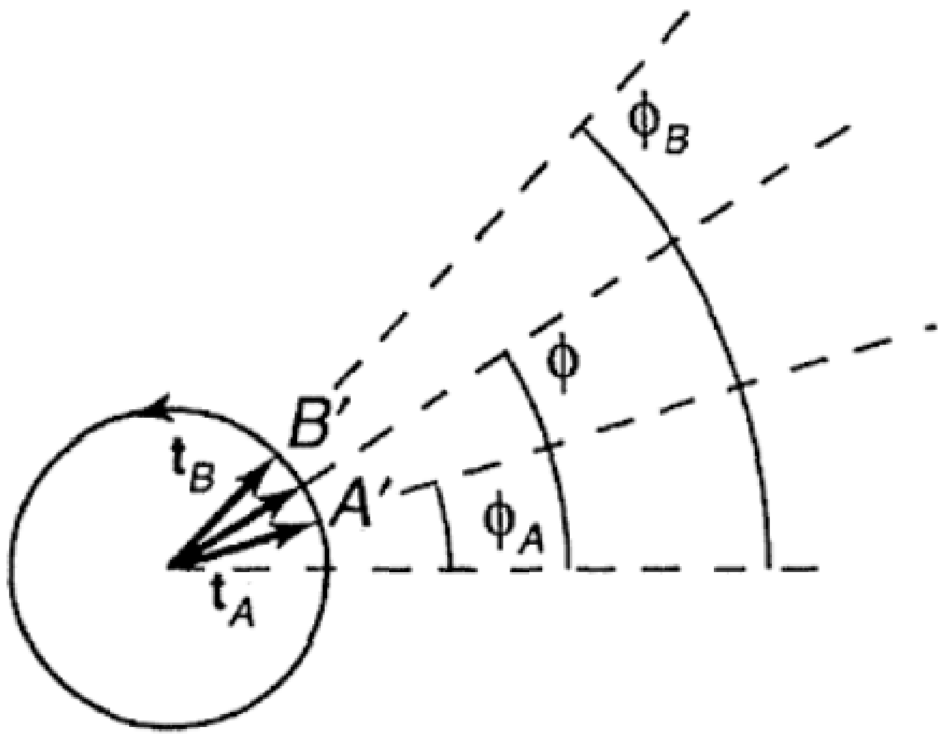
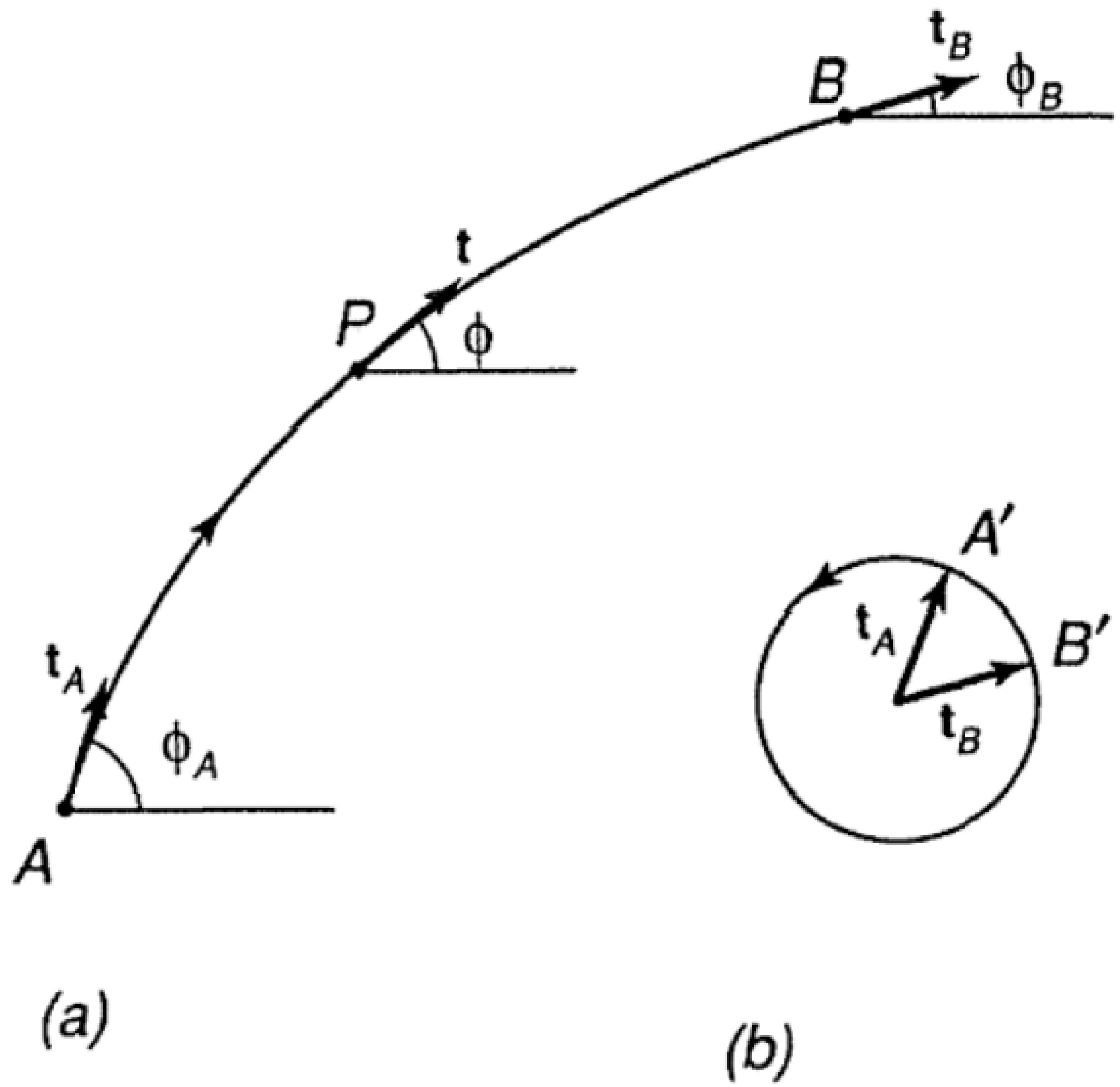
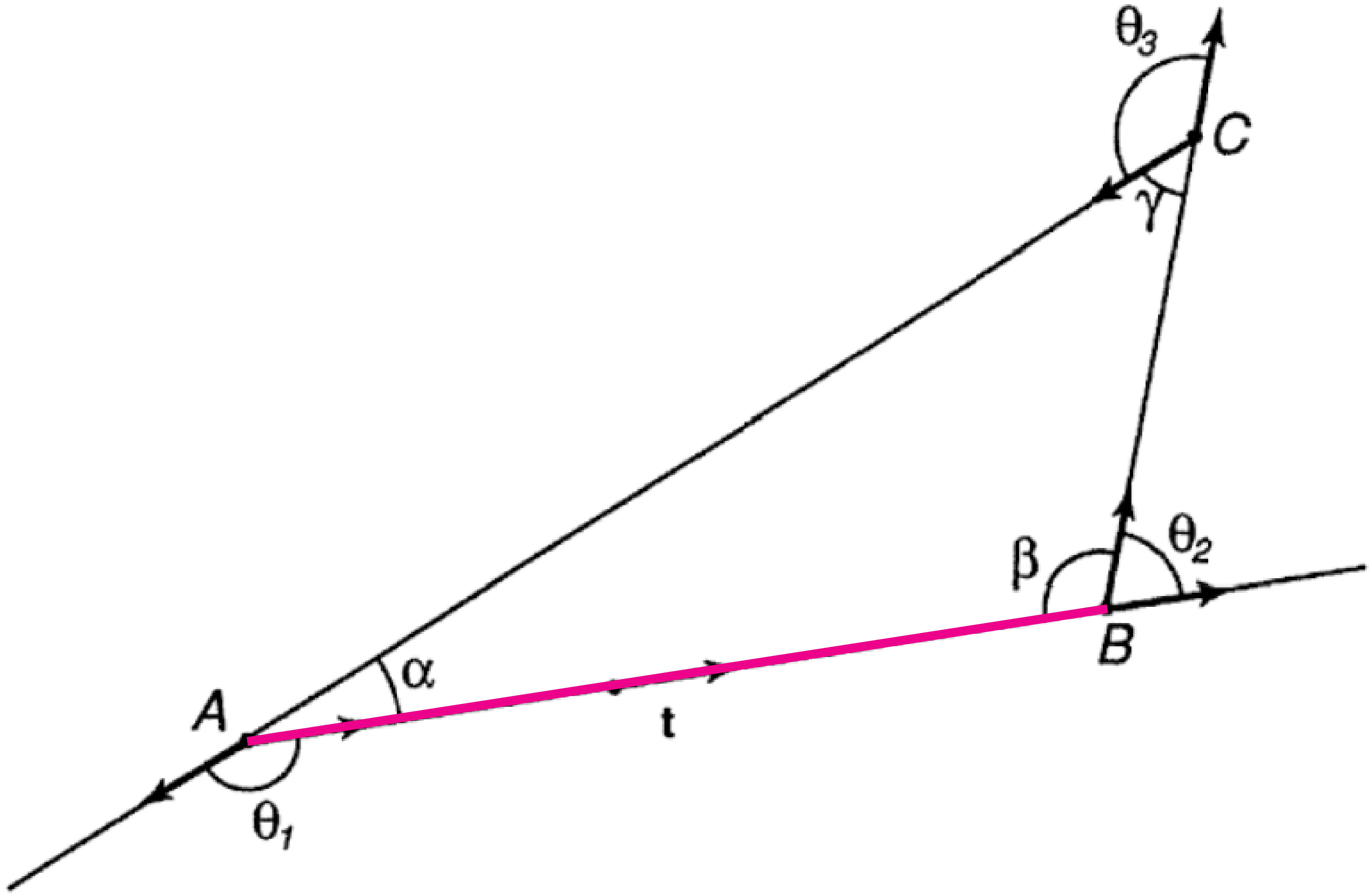
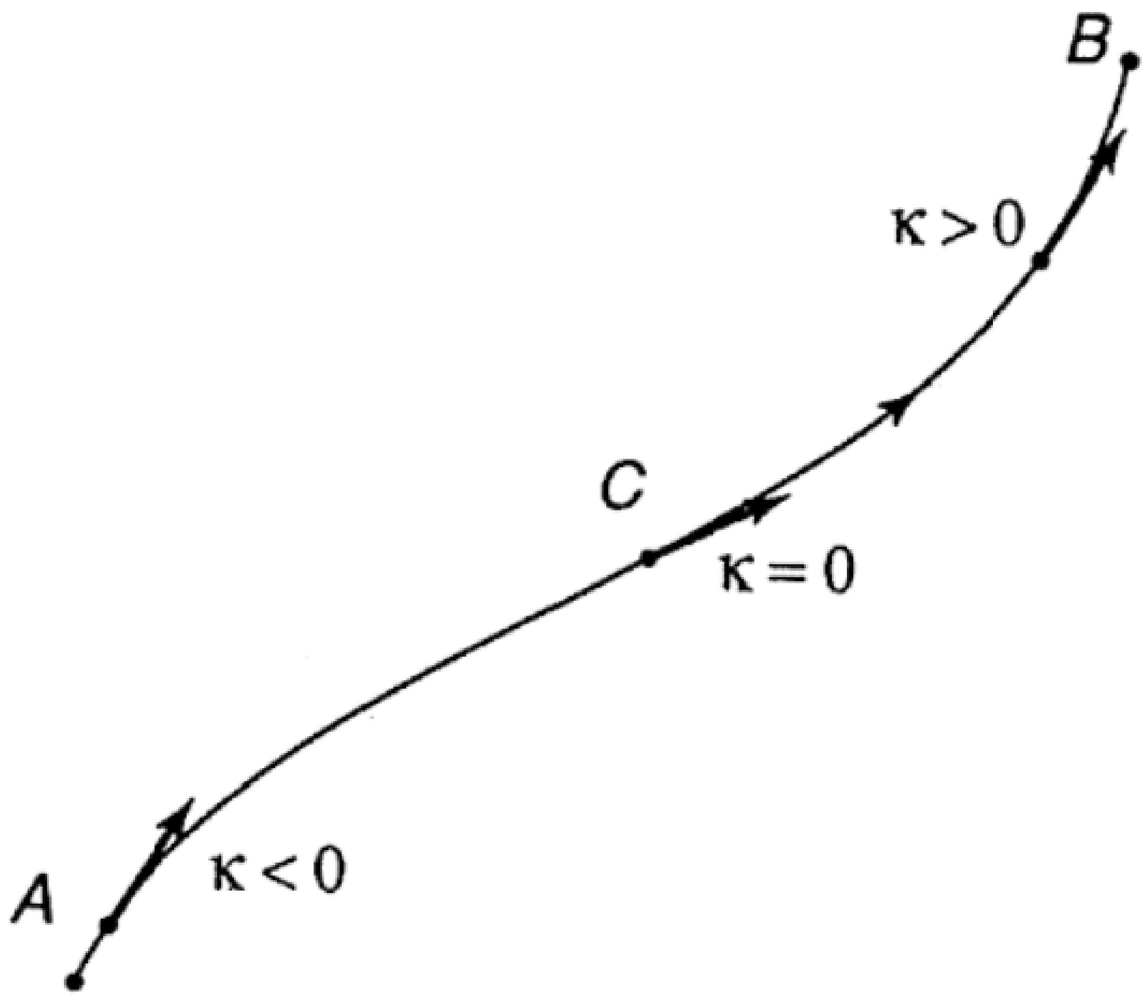


Figure 63 Auxiliary unit circle


t_B
B
ϕ_B
t
P
ϕ
t_A
ϕ_A
A
A′
t_A
B′
t_B
(a)
(b)


θ3
C
γ
β
θ2
B
α
A
t
θ1


B
κ > 0
C
κ = 0
A
κ < 0